Pure bosonic worldline path integral representation for fermionic determinants, non-Abelian Stokes theorem, and quasiclassical approximation in QCD

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Abstract

Simple bosonic path integral representation for path ordered exponent is derived. This representation is used, at first, to obtain new variant of non-Abelian Stokes theorem. Then new pure bosonic worldline path integral representations for fermionic determinant and Green functions are presented. Finally, applying stationary phase method, we get quasiclassical equations of motion in QCD.

1 Introduction

Elimination of fermionic degrees of freedom is very desirable in many problems of quantum field theory and elementary particle physics. First, in lattice gauge theories integration with respect to Grassmannian variables leads to serious complications of numerical simulations. The second, appearance of fermionic variables in functional integrals hampers the application of stationary phase method. As a consequence, one cannot also apply quasiclassical

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expansions to evaluation of functional integrals in theories with fermions except some especially simple cases.

Indeed, typical Green function can be written as

$$G(x_1, ..., x_n) = \int DAD\Psi D\bar{\Psi} e^{\frac{i}{\hbar}(S_{YM}(A)+S_{ferm}(\bar{\Psi},\Psi,A))}\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n) \qquad (1)$$

where $\mathcal{O}_i$ are some operators whereas S_{YM} and S_{ferm} are Yang-Mills and fermionic actions respectively. Quasiclassical approximation is defined, up to some subtleties, by stationary point equations

$$\frac{\delta S_{YM}}{\delta A} = [\text{the source of YM field}] \qquad (2)$$

But what must be written in R.H.S. of eq. (2) ? Of course, one cannot put

$$[\text{the source of YM field}] = \frac{\delta S_{ferm}}{\delta A} \qquad (3)$$

because S_{ferm} depends on Grassmanian variables. Moreover, without preliminary exception of fermionic variables one cannot write in R.H.S. of (2) nothing except zero. But this means that in zero approximation YM field can be considered as free. It seems inappropriate in all cases in which interaction is strong.

So for application of quasiclassical methods as well as for facilitation of numerical simulations on the lattice fermionic variables in functional integrals of the type (1) must be integrated out and result must be represented as functional integral with respect to only bosonic variables. In other words, the theory must be bosonized.

The problem of bosonization of fermionic theories has a long history. Most likely, the first example of bosonisation of fermionic theory was given by Schwinger in his famous paper [1] concerning full solution of massless QED_2. Then the problem of bosonisation was investigated by many authors, but more or less exhaustive solution was obtained only in two dimensional case and in some three dimensional models (see, for instance, papers [2] and references therein). In realistic four dimensional case only partial success was achieved (see, for instance, [3]). In fact in all proposed bosonization schemes in four dimensions it is necessary to evaluate (exactly or in some

approximation) fermionic determinant – but it is just the main problem that must be solved by means of bosonization.

To author's knowledge, the only exceptions are recent papers by Lusher [4] and Slavnov [5] (see also [6]). In Ref. [4] fermionic determinant on the finite lattice is represented as infinite some of bosonic determinants. In Ref. [5] fermionic determinant in D dimensions is expressed via bosonic one in $D+1$ dimensions. These approaches seem useful in lattice theories but they cannot be applied to investigation of quasiclassical approximation.

So hitherto no quite satisfactory representation for fermionic determinant in terms of bosonic fields is known, and at present paper we will develop another approach to bosonization. Namely, we will derive pure bosonic worldline path integral representation for fermionic determinants, Green functions and Wilson loops.

Worldline approach to quantum field theory also has very long history. It was originated many years ago in classical works by Feynman [7] and Schwinger [8] . The main idea of this approach is to represent fermionic determinants and fermionic Green functions as functional integral over trajectory of a single relativistic particle.

Let us consider, for instance, fermionic determinant for $SU(N)$ Yang-Mills theory in Euclidean space:

$$D \equiv \det(i\hat{\nabla} + im) = \det(i\hat{\nabla} + im)\gamma^5 \tag{4}$$

where $\hat{\nabla} = \gamma^\mu \nabla_\mu = \gamma^\mu(\partial_\mu - iA_\mu)$, $(\gamma^\mu)^\dagger = \gamma^\mu$, $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, $iA_\mu(x) \in su(N)$.

One can write:

$$\ln D = \frac{1}{2}\ln\det[(i\hat{\nabla} + im)\gamma^5]^2 = \frac{1}{2}\ln\det(-\nabla_\mu\nabla_\mu + \sigma^{\mu\nu}F_{\mu\nu} + m^2) \tag{5}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{6}$$

$$\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{7}$$

Further, eq. (5) can be written, up to inessential constant, as

$$\begin{aligned}\ln D &= \frac{1}{2}\mathrm{tr}\,\ln(-\nabla_\mu\nabla_\mu + \sigma^{\mu\nu}F_{\mu\nu} + m^2) \\ &= -\frac{1}{2}\int_0^\infty \frac{dT}{T}e^{-m^2T}\mathrm{tr}\,e^{-T(-\nabla_\mu\nabla_\mu+\sigma^{\mu\nu}F_{\mu\nu})}\end{aligned} \tag{8}$$

The integral (8) is divergent at $T = 0$ and must be regularized. One can use, for instance, ζ-function regularization:

$$\ln D = -\frac{1}{2}\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^\infty dT\; T^{s-1}e^{-m^2T}\mathrm{tr}\,e^{-T(-\nabla_\mu\nabla_\mu+\sigma^{\mu\nu}F_{\mu\nu})}\Big|_{s=0} \tag{9}$$

However, in what follows we shall write, for short, formal expression (8) for $\ln D$ bearing in mind any suitable regularization.

Further, trace in eq.(8) can be represented as functional integral:

$$\begin{aligned}&\mathrm{Tr}e^{-T(-\nabla_\mu\nabla_\mu+\sigma^{\mu\nu}F_{\mu\nu})} \\ = &\int_{PBC} Dq\;\mathrm{tr}\;\mathrm{P}\exp\left\{-\int_0^1 dt\left(\frac{\dot q^2}{4T} - i\dot q^\mu A_\mu(q) + T\sigma^{\mu\nu}F_{\mu\nu}(q)\right)\right\}\end{aligned} \tag{10}$$

Here PBC means 'periodic boundary conditions'. Path ordering in (10) corresponds to both spin and colour matrix structures.

Let us suppose for a moment that spin and colour degrees of freedom are absent in (10). (This is just the case of scalar electrodynamics). Then, integrating in (1) over fermionic fields and using for arising fermionic determinants and fermionic Green functions formulae of the type (8), (10), one obtains formulation of quantum field theory in terms of particles interacting with gauge field A. This is just result of classical work bu Feynman [7].

In realistic models, however, one must take into account spin and colour degrees of freedom. So, to develop worldline formulation of quantum field theory, it is necessary to represent the path ordered exponent in (10) as functional integral. In the case of QED this was done by Fradkin [9] in terms of fermionic path integral. This representation and their modifications were successfully used, in particular, for construction of derivative expansion in QED [10] and for investigation of complicated Feynman diagrams [11]. It is also intensively used for investigation of hidden supersymmetry in fermionic

theories (see, for example, [12] and references therein). Recently, D'Hoker and Gagne derived fermionic path integral representation for fermionic determinants for particles coupled with arbitrary tensor field [13].

In papers [8]-[13] fermionic path integral representations were derived and used only for spin degrees of freedom. For colour P-exponent in (10) fermionic path integral representation also can be derived (see, for example, [14]) though it seems not so elegant as one for spin P-exponent. Apropos, works [14] were, to author's knowledge, the first attempts to obtain non-perturbative information about QCD in worldline formalism.

But, as we already stated at the beginning of the present paper, there exist important problems for which pure bosonic worldline path integral representation for fermionic determinants and Green functions is very desirable. We see that for solution of the latter problem it is sufficient to derive bosonic worldline path integral representation for the trace of path ordered exponent

$$Z = \mathrm{tr}\ \mathrm{P}e^{i\int_0^1 dt B(t)} \tag{11}$$

with matrix $B(t) \in GL(N),\ B(0) = B(1)$. Indeed, substituting such representation with

$$B(t) = \dot{q}^\mu A_\mu(q(t)) - \sigma^{\mu\nu} F_{\mu\nu}(q(t)) \tag{12}$$

in (10), one obtains desired representation for fermionic determinant. As we will show later, bosonic path integral representation for Z allows also to obtain bosonic worldline path integral representation for fermionic Green functions.

Different bosonic path integral representations for Z in the case $B(t) \in SU(2)$ were proposed by several authors (see [15]-[18]). For more general case $B(t) \in su(N)$ the an analogous representation was pointed out in [19].

The typical result for $B \in SU(2)$ is

$$Z = \int DS(t) \exp\left\{\frac{i}{2}\int_0^1 dt \left[\mathrm{tr}\sigma^3 \left(S(t)B(t)S^\dagger(t) + iS(t)\dot{S}^\dagger(t)\right)\right]\right\} \tag{13}$$

The integration in (13) is carried out over all trajectories in the group $SU(2),\ S \in SU(2),\ \sigma^3$ is a Pauli matrix.

The last result has two disadvantages. First, in any parametrization of $SU(2)$ the "action" in (13) is rather complicated non-polinomial function.

This hampers the usage of (13) in practice. The second, it appears that integral (13) is ill-defined and needs insertion of some regulators in the "action" (see the discussion of this point in [16, 19] and in more recent paper [20]).

Recently Dyakonov and Petrov found much more elegant formula for path ordered exponent. Namely, for the case when Z is Wilson loop and gauge group is $SU(2)$, they derived from (13) the following expression for Z:

$$\begin{aligned} Z &= \mathrm{tr}\,\mathrm{P}\, e^{i\oint_\gamma dx^\mu A_\mu(x)} \\ &= \int Dn(x) \prod_{x\in\Sigma} (n^a(x)n^a(x)-1) \\ &\quad \exp\left\{\frac{i}{4}\int_\Sigma dx^\mu \wedge dx^\nu \left[-F^a_{\mu\nu}n^a + \varepsilon^{abc}n^a D_\mu n^b D_\nu n^c\right]\right\} \end{aligned} \tag{14}$$

where Σ is two dimensional surface spanned on contour γ, $a = 1, 2, 3$, D^μ – covariant derivative. This formula can be considered as non-Abelian variant of Stokes theorem. We shall continue the discussion of this result in section 4.

In the present paper we will derive alternative bosonic path integral representation for Z in the general case $B \in su(N)$. The "action" in this representation is quadratic and so it is much more simpler then one in (13) and (14). This derivation is presented in the section 2. In the section 3 we check the representation obtained by direct evaluation of the functional integral that defines Z. In fact, we give alternative proof of results obtained in the section 2. In the section 4 we derive an non-Abelian analog of Stokes theorem and compare our results with those due to Dyakonov and Petrov. In the section 5 we derive bosonic worldline path integral representation for fermionic determinant and Green functions in Euclidean QCD. In the section 6 we get analogous results for Minkowski space and then, applying stationary phase method, derive quasiclassical equations of motion in QCD. In the last section we summarize our results and discuss perspectives of future investigations. In two appendixes we derive some auxiliary formulae that are used in the main text of the paper.

2 Bosonic path integral representation for the trace of path ordered exponent

Let $N \times N$ matrix $U(t)$, $0 \le t \le 1$, be defined by equations

$$
\begin{aligned}
\frac{dU}{dt} &= iB(t)U(t) \\
U(0) &= I_N
\end{aligned}
\tag{15}
$$

where I_N is $N \times N$ unit matrix. Then, obviously,

$$
Z = \operatorname{tr} U(1) \tag{16}
$$

First, we consider the case $B \in su(N)$, that is

$$
B^\dagger = B, \quad \operatorname{tr} B = 0 \tag{17}
$$

Let us consider an operator

$$
\hat{B} = a_r^\dagger B_s^r a^s \tag{18}
$$

that acts in some Fock space $\mathcal{F}$. Operators $a_r^\dagger$ and a^s in (18) are usual *bosonic* creation and annihilation ones, $[a^r, a_s^\dagger] = \delta_s^r$.

Let $\mathcal{H}_n$ be n-particle subspace of $\mathcal{F}$, Π_n is orthogonal projector on $\mathcal{H}_n$. Then

$$
Z = \operatorname{tr} \Pi_1 \, \mathrm{P} \, e^{i \int_0^1 dt \, \hat{B}(t)} \tag{19}
$$

Indeed, if $\hat{N} = a_r^\dagger a^r$, then

$$
\left[\hat{B}, \hat{N}\right] = 0 \tag{20}
$$

So $\hat{B}\mathcal{H}_n \subset \mathcal{H}_n$ and in one dimensional subspace $\mathcal{H}_1$ the operator $\hat{B}$ can be identified with the matrix B via relation:

$$
\hat{B} a_r^\dagger \phi^r |0> = a_r^\dagger (B_s^r \phi^s)|0> \tag{21}
$$

Projector Π_1 can be represented as

$$
\begin{aligned}
\Pi_1 &= \int_{1-\varepsilon}^{1+\varepsilon} d\lambda \, \delta(\hat{N} - \lambda) \\
&= \int_{1-\varepsilon}^{1+\varepsilon} d\lambda \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \, e^{-i\lambda\eta + i\hat{N}\eta}
\end{aligned}
\tag{22}
$$

where $0 < \varepsilon < 1$.

So

$$
\begin{aligned}
Z &= \lim_{\delta\to+0} \operatorname{tr} \Pi_1 e^{-\delta\hat N}\, \mathrm{P}\, e^{i\int_0^1 dt\, \hat B(t)} \\
&= \lim_{\delta\to+0} \int_{1-\varepsilon}^{1+\varepsilon} d\lambda \int\limits_{-\infty}^{\infty} \frac{d\eta}{2\pi} \operatorname{tr} e^{-i\lambda\eta+(i\eta-\delta)\hat N} e^{i\int_0^1 dt\, \hat B(t)} \\
&= \lim_{\delta\to+0} \int_{1-\varepsilon}^{1+\varepsilon} d\lambda \int\limits_{-\infty}^{\infty} \frac{d\eta}{2\pi} \operatorname{tr} e^{-i\lambda\eta} \operatorname{tr}\ \mathrm{P}\, e^{i\int_0^1 dt(\hat B(t)+(\eta+i\delta)\hat N)}
\end{aligned} \tag{23}
$$

The latter equality is valid due to (20).

The trace of ordered exponent in (23) can be represented as functional integral. In our case its explicit form essentially depends on some subtleties in its definition. So let me remind in a few words the general construction of functional integral. For more detailed discussion see, for instance, [22].

Let

$$\check Z = \operatorname{tr}\ \mathrm{P}\, e^{\int_0^1 dt\, \hat H} \tag{24}$$

where $\hat H = \hat H(a^\dagger, a; t)$ is an operator in Fock space $\mathcal{F}$.

In our case

$$\hat H = (i\eta - \delta)\hat N + i\hat B \tag{25}$$

Equivalently, we can write

$$\hat H = \hat H(\hat p, \hat q; t) \tag{26}$$

$$\hat p = \frac{a + a^\dagger}{\sqrt 2}, \qquad \hat q = \frac{a - a^\dagger}{i\sqrt 2} \tag{27}$$

Let $H = H(p, q; t) \equiv H(\bar z, z; t)$ be Weyl, or normal, or any other symbol of the operator $\hat H$,

$$z = \frac{p - iq}{\sqrt 2}, \qquad \bar z = \frac{p + iq}{\sqrt 2} \tag{28}$$

Further, let $*$ be the operation that represent the multiplication of operators in the language of symbols. This means that if $\hat{K} = \hat{K}(\hat{p}, \hat{q})$ and $\hat{L} = \hat{L}(\hat{p}, \hat{q})$ are some operators with symbols $K = K(p, q)$ and $L = L(p, q)$ and $\hat{M} = \hat{K}\hat{L}$ then

$$M(p, q) = (K * L)(p, q) \tag{29}$$

In this terms

$$\check{Z} = \lim_{n\to\infty} \int dpdq \left(e^{\frac{1}{n}H(\cdot,\cdot;0)} * e^{\frac{1}{n}H(\cdot,\cdot;\frac{1}{n})} * \ldots * e^{\frac{1}{n}H(\cdot,\cdot;\frac{n-1}{n})} \right)(p, q) \tag{30}$$

The last formula can be rewritten as functional integral. Its concrete form depends on the choice of a kind of symbols used in (30). In particular, for Weyl symbols

$$\check{Z} = \mathcal{N} \int_{PBC} DpDq\, e^{\int_0^1 dt\ (ip(t)\dot{q}(t) + H_W(p(t),q(t);t))} \tag{31}$$

whereas for normal symbols

$$\check{Z} = \lim_{\epsilon\to+0} \mathcal{N}' \int_{PBC} DzD\bar{z}\, e^{\int_0^1 dt(\bar{z}(t)\dot{z}(t) + H_{norm}(\bar{z}(t), z(t+\epsilon))} \tag{32}$$

Here PBC means 'periodic boundary conditions', z and $\bar{z}$ are independent complex variables, and $\mathcal{N}, \mathcal{N}'$ are normalization constants (that we will usually omit in what follows).

In our case

$$\begin{aligned} H_W(p, q; t) &= H_W(z^\dagger, z; t) \\ &= (i\eta - \delta)z^\dagger z + iz^\dagger Bz - \frac{N}{2}(i\eta - \delta) \end{aligned} \tag{33}$$

where p, q are connected with z by formulae (28), and

$$H_{norm}(\hat{z}, z) = (i\eta - \delta)z^\dagger z + iz^\dagger Bz \tag{34}$$

Formulae (31), (32) correspond to standard sign conventions. However, for our purposes it is convenient to change variables

$$z(t) \to z(1 - t)$$

Then formulae (31), (32) with symbols (33), (34) can be rewritten as

$$\begin{aligned} \check{Z} &= \int_{PBC} D^2 z \, \exp\left\{ i \int_0^1 dt \left[i z^\dagger(t) \dot{z}(t) + z^\dagger(t) B(t) z(t) \right.\right. \\ &+ \left.\left. (\eta + i\delta) z^\dagger(t) z(t) - \frac{N}{2}(\eta + i\delta) \right] \right\} \end{aligned} \tag{35}$$

where $D^2 z \equiv D(\mathrm{Re} z) D(\mathrm{Im} z)$ and

$$\begin{aligned} \check{Z} &= \lim_{\epsilon \to +0} \int Dz D\bar{z} \, \exp\left\{ i \int_0^1 dt \left[i \bar{z}(t) \dot{z}(t) + \bar{z}(t) B(t) z(t - \epsilon) \right.\right. \\ & \left.\left. +(\eta + i\delta) \bar{z}(t) z(t - \epsilon) \right] \right\} \end{aligned} \tag{36}$$

respectively.

There are several essential differences between formulae (35) and (36). First, in (35) z and $z^\dagger$ are complex conjugated variables whereas in (36) $\bar{z}$ and z are independent. The second, the last term in the "action" in (35) is absent in (36). The third, there is the shift of "time" variable in the last two terms in the "action" in (36) that is absent in (35). In the next section we will show by explicit calculations that this shift just compensates the absence of the term

$$-\frac{N}{2}(\eta + i\delta)$$

in (36). Finally, it is worth to note that the limit in (36) must be evaluated *after* functional integration in (36) because this two operations don't commute. It will be confirmed by explicit calculations in the section 3.

Substituting (35) and (36) in (23) and evaluating limits $\varepsilon \to 0$, $\delta \to 0$ (bat not the limit $\epsilon \to +0$!), ones obtains, respectively,

$$Z = \int_{PBC} D^2 z \int\limits_{-\infty}^{\infty} \frac{d\eta}{2\pi} \, e^{i \int_0^1 dt (i z^\dagger \dot{z} + z^\dagger B z + \eta (z^\dagger z - 1 - \frac{N}{2})} \tag{37}$$

and

$$Z = \lim_{\epsilon\to+0}\int_{PBC} Dz D\bar{z} \int\limits_{-\infty}^{\infty} \frac{d\eta}{2\pi} \exp\Big\{ i\int_0^1 dt\, i\bar{z}\dot{z} + \bar{z}Be^{-\epsilon\frac{d}{dt}}z$$
$$+ \quad \eta(\bar{z}e^{-\epsilon\frac{d}{dt}}z - 1)\Big\} \tag{38}$$

We won't try to justify here the validity of limiting procedure $\varepsilon \to 0$, $\delta \to 0$ because in the next section we will check formulae (37), (38) by direct calculation.

In (37) one can integrate with respect to η:

$$Z = \int_{PBC} D^2z\, \delta(\int_0^1 dt\, z^\dagger z - 1 - \frac{N}{2}) e^{i\int_0^1 dt(iz^\dagger\dot{z} + z^\dagger Bz)} \tag{39}$$

But we cannot obtain δ-function by integration with respect to η in (38) because $\bar{z}$ and z in (38) are independent complex variables.

One can also get another useful form of the representation (39), namely

$$Z = \int_{PBC} D^2z \prod_t \delta(z^\dagger(t)z(t) - 1 - \frac{N}{2}) e^{i\int_0^1 dt(iz^\dagger\dot{z} + z^\dagger Bz)} \tag{40}$$

In what follows, we will sometimes omit symbol $\prod$ in formulae of the type (40).

To get formula (40), it is sufficient to insert projectors Π_1 represented in the form (22) between each pair of adjacent factors in (30) and to repeat all calculations that have led us to representation (39).

The same arguments allow also to obtain an analogous variant of (38):

$$Z = \lim_{\epsilon\to+0}\int_{PBC} Dz(t)D\bar{z}(t)D\eta(t) \exp\{\, i\int_0^1 dt\, i\bar{z}(t)\dot{z}(t) + \bar{z}(t)B(t)e^{-\epsilon\frac{d}{dt}}z(t)$$
$$+ \quad \eta(t)(\bar{z}(t)e^{-\epsilon\frac{d}{dt}}z(t) - 1)\,\} \tag{41}$$

Thus we have derived four different representations (38)-(41) for one quantity Z. They all appear to be useful in quantum field theory.

We have got (38)-(41) assuming that $B \in su(N)$. But these representation remains valid for every trace free matrix by virtue of analytical continuation. Representation for $B \in GL(N)$ can be obtained by extracting of the trace part of the matrix B at the beginning of calculations. Indeed, if

$$B = B' + \frac{1}{N} I_N \,\mathrm{tr}\, B$$

then $\mathrm{tr}\, B' = 0$ and

$$\mathrm{tr}\ \mathrm{P}\, e^{i\int_0^1 dt\, B} = e^{\frac{i}{N}\int_0^1 dt\,\mathrm{tr}\, B}\ \mathrm{tr}\ \mathrm{P}\, e^{i\int_0^1 dt\, B'} \tag{42}$$

Another representation for matrices with non-zero trace will be given in the next section (see eq. (65)).

Finally, let us derive a kind of representation (40) for the case

$$B(t) = \sum_{j=1}^{M} C_j(t) \otimes D_j(t) \tag{43}$$

where C_j and D_j are $N_1 \times N_1$ and $N_2 \times N_2$ matrices. Let $a_{(i)}^{r_i},\quad a_{(i) r_i}^{\dagger},\quad i = 1, 2\quad r_i = 1, 2, \ldots, N_i$ be two sets of annihilation and creation operators that act in some Fock space, and $\Pi_{1\otimes 1}$ is a projector on the space

$$\mathcal{H}_{1\otimes 1} = \left\{ \phi^{r_1 r_2} a_{(1) r_1}^{\dagger} a_{(2) r_2}^{\dagger} |0> \right\} \tag{44}$$

One can write

$$\begin{aligned} &\mathrm{tr}\ \mathrm{P}\, e^{i\int_0^1 dt\, B} \\ &\quad = \mathrm{tr}\, \Pi_{1\otimes 1}\ \mathrm{P}\ \exp\left\{ i \int_0^1 dt \sum_{j=1}^{M} (a_{(1)}^{\dagger} C_j a_{(1)})(a_{(2)}^{\dagger} D_j a_{(2)}) \right\} \end{aligned} \tag{45}$$

The last formula ia an analog of (19). Then, repeating the calculations that have been done for derivation of eq. (39) from eq. (19), one obtains:

$$\begin{aligned} \mathrm{tr}\ \mathrm{P}\, e^{i\int_0^1 dt\, B} \quad = \quad & \int D^2 z_1 D^2 z_2 \prod_{i=1}^{2} \delta(z_{(i)}^{\dagger} z_{(i)} - 1 - \frac{N}{2}) \\ & \exp\left\{ i \int_0^1 dt \left[i \sum_{i=1}^{2} z_{(i)}^{\dagger} \dot{z}_{(i)} \right.\right. \\ & \left.\left. + \sum_{j=1}^{M} (z_{(1)}^{\dagger} C_j z_{(1)} - \frac{1}{2}\,\mathrm{tr}\, C_j)(z_{(2)}^{\dagger} D_j z_{(2)} - \frac{1}{2}\,\mathrm{tr}\, D_j) \right] \right\} \end{aligned} \tag{46}$$

Similar arguments allow to obtain the analog of eqs. (38), (40), and (41) for an matrix B defined by eq. (43).

3 Alternative proof of bosonic path integral representation for the trace of path ordered exponent

At first, we will evaluate the integral (39) assuming that $B \in su(N)$. One can write:

$$
\begin{aligned}
Z &= \lim_{\epsilon \to +0} \int_{PBC} D^2 z\, \delta\Big(\int_0^1 dt\, z^\dagger z - 1 - \frac{N}{2}\Big) e^{-\epsilon \int_0^1 dt\, z^\dagger z} e^{i \int_0^1 dt (i z^\dagger \dot{z} + z^\dagger B z)} \\
&= \lim_{\epsilon \to +0} \int\limits_{-\infty}^{\infty} \frac{d\eta}{2\pi}\, e^{-i\eta\left(1+\frac{N}{2}\right)} \int_{PBC} D^2 z\, e^{\int_0^1 dt\, z^\dagger \left(-\frac{d}{dt} + iB + i\eta - \epsilon\right) z}
\end{aligned} \tag{47}
$$

Performing functional integration, one gets:

$$
Z = \lim_{\epsilon \to +0} \int\limits_{-\infty}^{\infty} \frac{d\eta}{2\pi} \frac{e^{-i\eta\left(1+\frac{N}{2}\right)}}{\det\left(-\frac{d}{dt} + i(B+\eta) - \epsilon\right)_{PBC}} \tag{48}
$$

To evaluate integral in (48), let us consider eigenvalue problem

$$
\left(-\frac{d}{dt} + i(B+\eta) - \epsilon\right) \phi(t) = \lambda \phi(t) \tag{49}
$$

$$
\phi(0) = \phi(1) \tag{50}
$$

The general solution of eq. (49) is

$$
\phi = e^{(-\lambda + i\eta - \epsilon)t} U(t) \chi \tag{51}
$$

where $U(t)$ is a solution of (15) and vector χ doesn't depend on t.

Let $e^{i\alpha_r}$ and $\xi_r, \quad r = 1, \ldots, N$ be eigenvalues and eigenvectors of the matrix $U(1)$:

$$
U(1)\xi_r = e^{i\alpha_r} \xi_r \tag{52}
$$

One notes that $U(1) \in SU(N)$ and so α_r are real and

$$
\sum_{r=1}^{N} \alpha_r = 0 \tag{53}
$$

Further, to satisfy (50) we must put

$$\lambda \equiv \lambda_{rn} = -\epsilon + i\eta + i\alpha_r + 2\pi i n, \quad n = 0, \pm 1, \dots \tag{54}$$

and $\chi = \xi_r$. So

$$\begin{aligned}
\det\left(-\frac{d}{dt} + i(B+\eta) - \epsilon\right)_{PBC} &= \prod_{r=1}^{N} \prod_{n=-\infty}^{\infty} (2\pi i n + i\alpha_r + i\eta - \epsilon) \\
= \quad & \left[-i \prod_{n=-\infty}^{\infty} (2\pi i n)\right]^r \prod_{r=1}^{N} (\alpha_r + \eta + i\epsilon) \prod_{n\neq 0} \left(1 + \frac{\alpha_r + \eta + i\epsilon}{2\pi n}\right) \\
= \quad & \left[-i \prod_{n=-\infty}^{\infty} (2\pi i n)\right]^r \prod_{r=1}^{N} (\alpha_r + \eta + i\epsilon) \prod_{n=1}^{\infty} \left(1 - \frac{(\alpha_r + \eta + i\epsilon)^2}{4\pi^2 n^2}\right)
\end{aligned} \tag{55}$$

Then, omitting irrelevant infinite constant and using well-known formula

$$\prod_{n=1}^{\infty} \left(1 - \frac{a^2}{n^2}\right) = \frac{1}{\pi a} \sin \pi a \tag{56}$$

one gets

$$\det\left(-\frac{d}{dt} + i(B+\eta) - \epsilon\right)_{PBC} = \prod_{r=1}^{N} \sin\left(\frac{\alpha_r + \eta + i\epsilon}{2}\right) \tag{57}$$

Substituting (57) in (48), one obtains contour integral

$$Z = \lim_{\epsilon \to +0} \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \frac{e^{-i\eta\left(1+\frac{N}{2}\right)}}{\prod_{r=1}^{N} \sin\left(\frac{\alpha_r+\eta+i\epsilon}{2}\right)} \tag{58}$$

One can close the contour of integration in (58) in the lower half plane and represent Z as the sum of residues in the poles

$$\eta_{rn} = -\alpha_r - i\epsilon + 2\pi n \tag{59}$$

The contribution of the pole with given r and n is

$$2\pi i (-1)^{n+1} \frac{e^{i(\alpha_r + i\epsilon)}}{\prod\limits_{\substack{1\leq s\leq N \\ s\neq r}} \sin\left(\frac{\alpha_s - \alpha_r}{2}\right)} \tag{60}$$

So, omitting again inessential constant, one gets:

$$
\begin{aligned}
Z &= \lim_{\epsilon\to+0} \sum_{r,n} \left(\begin{array}{c} \text{contribution of the residue in} \\ \eta_{rn} = -\alpha_r - i\epsilon + 2\pi n \end{array} \right) \\
&= \sum_{r=1}^{N} \frac{e^{i\alpha_r(1+\frac{N}{2})}}{\prod\limits_{\substack{s=1 \\ s\neq r}}^{N} \sin\left(\frac{\alpha_s-\alpha_r}{2}\right)} \qquad (61)
\end{aligned}
$$

Finally, using elementary but rather non-obvious identity

$$
\sum_{r=1}^{N} \frac{e^{i\alpha_r(1+\frac{N}{2})}}{\prod\limits_{\substack{s=1 \\ s\neq r}}^{N} \sin\left(\frac{\alpha_s-\alpha_r}{2}\right)} = (-2i)^{N-1} \left(e^{\frac{i}{2}\sum\limits_{r=1}^{N} \alpha_r} \right) \sum_{r=1}^{N} e^{i\alpha_r} \qquad (62)
$$

(that is valid for any complex numbers α_r; see Appendix A for proof), one obtains:

$$
Z = \sum_{r=1}^{N} e^{i\alpha_r} = \mathrm{tr}\, U(1) = \mathrm{tr}\ \mathrm{P}\, e^{i\int_0^1 dt\, B(t)} \qquad (63)
$$

This proves the representation (39).

Throughout the proof we didn't control inessential numerical normalization factors arising in front of functional integrals, determinants, etc. They can be easily reconstructed from normalization condition

$$
Z\Big|_{B=0} = N \qquad (64)
$$

The representation (39) is valid for traceless matrix B. In general case the following representation is valid

$$
Z = e^{-\frac{i}{2}\int_0^1 dt\,\mathrm{tr}\, B(t)} \int D^2 z\, \delta\left(\int_0^1 dt\, z^\dagger z - 1 - \frac{N}{2} \right) e^{i\int_0^1 dt (iz^\dagger \dot{z} + z^\dagger B z)} \qquad (65)
$$

This representation is alternative to one given by eq. (42).

To prove (65), it is sufficient to repeat the proof given for representation (39) but without using the condition (53) . One can trace the cancellation of the pre-integral factor in (65) and contribution of the factor

$$e^{\frac{i}{2}\sum\limits_{r=1}^{N}\alpha_r}$$

in (62).

Now let us check the representation (40). Representing δ-function in (40) as

$$\prod_t \delta\left(z^\dagger z - 1 - \frac{N}{2}\right) = \int D\eta \, e^{i\int_0^1 dt\left(z^\dagger z - 1 - \frac{N}{2}\right)\eta} \tag{66}$$

and integrating over $z^\dagger, z$, one obtains the analog of (48):

$$Z = \lim_{\epsilon\to+0}\int D\eta(t)\,\frac{e^{-i\left(1+\frac{N}{2}\right)\int_0^1 dt\,\eta(t)}}{\det\left(-\frac{d}{dt} + i(B(t)+\eta(t)) - \epsilon\right)_{PBC}} \tag{67}$$

The eigenvalue problem

$$\left(-\frac{d}{dt} + i(B(t)+\eta(t)) - \epsilon\right)\phi(t) = \lambda\phi(t) \tag{68}$$

$$\phi(0) = \phi(1) \tag{69}$$

has the solution

$$\lambda \equiv \lambda_{rn} = -\epsilon + i\alpha_r + i\int_0^1 dt\,\eta(t) + 2\pi i n, \quad n = 0, \pm 1, \ldots \tag{70}$$

where numbers α_r are defined by equation (52). So determinant in (67) depends on $\eta(t)$ only via

$$\eta = \int_0^1 dt\,\eta(t) \tag{71}$$

Therefore, if one introduce new variables η and

$$\eta_n = \int_0^1 dt\, e^{2\pi i n t}\eta(t), \quad n \neq 0 \tag{72}$$

instead of $\eta(t)$ in (67), one finds that integration with respect to η_n gives only inessential constant and so the integral (67) transforms just in the integral (48). But this means that we reduced the proof of representation (40) to one of the representation (39) which we have already proved.

Finally, let us prove the validity of the representations (38) and (41). The proof can be reduced again to one of the representation (39) by means of the formula

$$\lim_{\epsilon\to+0}\det\left(-\frac{d}{dt}+iBe^{-\epsilon\frac{d}{dt}}\right)=e^{\frac{i}{2}\int_0^1 dt\,\mathrm{tr}\,B(t)}\det\left(-\frac{d}{dt}+iB(t)\right) \tag{73}$$

that is valid for any $N\times N$ matrix $B(t)$ (the proof of (73) is given in Appendix B). Indeed, evaluating the integral with respect to $\bar{z}$, z in (38), one obtains, after inserting of the factor $\exp\left(-\delta\int_0^1 dt\,z^\dagger(t)z(t+\epsilon)\right)$,

$$Z=\lim_{\delta\to+0}\lim_{\epsilon\to+0}\int\limits_{-\infty}^{\infty}\frac{d\eta}{2\pi}\,\frac{e^{-i\eta}}{\det\left(-\frac{d}{dt}+i(B+\eta+i\delta)e^{-\epsilon\frac{d}{dt}}\right)_{PBC}} \tag{74}$$

and further application of (73) reduce (74) to (48). (Remind, that $\mathrm{tr}\,B=0$ by assumption.)

The same arguments allow to check the representation(41).

Thus we have checked all four representations for the trace of path ordered exponent derived in the section 2. We see the proofs given in the present section are very unlike ones given in the section 2. This can be considered as strong confirmation of the validity of the representations obtained.

4 A variant of non-Abelian Stokes theorem

Analogs of Stokes theorem for Wilson loop

$$Z=\mathrm{tr}\ \mathrm{P}\,e^{i\oint_\gamma dx^\mu\,A_\mu(x)} \tag{75}$$

were proposed by several authors [23]. In this works Wilson loop is expressed via some area integral over surface spanned on γ with rather complicated path ordered prescriptions. Recently Dyakonov and Petrov derived another, very smart variant of non-Abelian Stokes theorem. Their result we already cited (see (14)).

In this section we will prove a new variant of non-Abelian Stokes theorem for $A_\mu\in su(N)$ that is similar to one given by Dyakonov and Petrov. For the case $N=2$ we will be able to derive from our results the Dyakonov-Petrov's

formula (14) but in slightly corrected form. The little discrepancy between our results and those by Dyakonov and Petrov arises, most likely, because of some subtleties in the definition of the corresponding functional integrals.

Let the path γ in (75) be parametrized as

$$\gamma = \{x^\mu = q^\mu(t),\ 0 \le t \le 1,\ q^\mu(0) = q^\mu(t)\} \tag{76}$$

Then

$$Z = \mathrm{tr}\ \mathrm{P}\, e^{i\int_0^1 dt\, \dot{q}^\mu(t) A_\mu(q(t))} \tag{77}$$

and we can apply the representation (40):

$$Z = \int D^2\psi \delta(z^\dagger z - 1 - \frac{N}{2}) e^{\int_0^1 dt\, z^\dagger\left(-\frac{d}{dt} + i\dot{q}A\right)z} \tag{78}$$

Let $\xi^r = \xi^r(x),\ r = 1, \dots, N$ be any field such that

$$\xi^r(q(t)) = z^r(t) \tag{79}$$

The formula (78) can be rewritten as

$$Z = \int D^2\xi \prod_{x\in\gamma} \delta\left(\xi^\dagger(x)\xi(x) - 1 - \frac{N}{2}\right) e^{-\oint_\gamma dx^\mu\, \xi^\dagger D_\mu \xi(x)} \tag{80}$$

where $D_\mu = \partial_\mu - iA_\mu$ is the usual covariant derivative. Further, applying the classical Stokes theorem, one can easy to prove that

$$\oint_\gamma dx^\mu\, \xi^\dagger D_\mu \xi(x) = \int_\Sigma dx^\mu \wedge dx^\nu \left(D_\mu \xi^\dagger D_\nu \xi - \frac{i}{2}\xi^\dagger F_{\mu\nu}\xi\right) \tag{81}$$

where Σ is any surface for which $\partial\Sigma = \gamma$.

Substituting (81) in (80), we get our variant of non-Abelian Stokes theorem:

$$\begin{aligned} Z \equiv\ & \mathrm{tr}\ \mathrm{P}\, e^{i\oint_\gamma dx^\mu\, A_\mu(x)} \\ =\ & \int D^2\xi \prod_{x\in\gamma} \delta\left(\xi^\dagger(x)\xi(x) - 1 - \frac{N}{2}\right) e^{\int_\Sigma dx^\mu \wedge dx^\nu \left(-D_\mu \xi^\dagger D_\nu \xi + \frac{i}{2}\xi^\dagger F_{\mu\nu}\xi\right)} \end{aligned} \tag{82}$$

Now let us transform eq. (82) into the form that would be similar to eq. (14).

Let λ^a, $a = 1, \ldots, N^2 - 1$ be Hermitian generators of $SU(N)$ in the fundamental representation. They can be normed as

$$\mathrm{tr}\, \lambda^a \lambda^b = 2\delta^{ab} \tag{83}$$

and satisfy equations

$$[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c \tag{84}$$

$$[\lambda^a, \lambda^b]_+ = \frac{4}{N}\delta^{ab} I_N + 2d^{abc}\lambda^c \tag{85}$$

One notes that

$$A_\mu = \frac{\lambda^a}{2} A^a_\mu, \quad F_{\mu\nu} = \frac{\lambda^a}{2} F^a_{\mu\nu} = \frac{\lambda^a}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu) \tag{86}$$

Matrices λ^a also obey "Fierz" identities:

$$\frac{1}{2}\lambda^a_{ij}\lambda^a_{kl} + \frac{1}{N}\delta_{ij}\delta_{kl} = \delta_{il}\delta_{kj} \tag{87}$$

$$if^{abc}\lambda^a_{ij}\lambda^b_{kl}\lambda^c_{mn} = 2(\delta_{il}\delta_{mj}\delta_{km} - \delta_{in}\delta_{jk}\delta_{ml}) \tag{88}$$

One introduces new variables

$$I^a(x) = \xi^\dagger(x)\lambda^a\xi(x) \tag{89}$$

Variables I^a are not independent. Using "Fierz" identity (87), formula (89) can be represented as

$$\frac{1}{\xi^\dagger\xi}(I^a\lambda^a)_{ij} + \frac{1}{N}\delta_{ij} = \xi_i\xi^\dagger_j\frac{1}{(\xi^\dagger\xi)} \tag{90}$$

Let $I \equiv I^a\lambda^a$, $\xi^\dagger\xi \equiv c$. One notes that $c = 1 + \frac{N}{2}$ on the surface of integration in (82).

The matrix I can be represented in the form

$$I = U diag(a_1, \ldots, a_N)U^\dagger \tag{91}$$

where

$$\sum_{i=1}^{N} a_i = 0, \quad U \in SU(N) \tag{92}$$

The matrix in R.H.S. of eq. (90) is a projector on the one dimensional subspace. This means that eigenvalues of the matrix in the L.H.S. of eq. (90) are all equal to zero except only one that is equal to 1. So, up to renumbering of eigenvalues,

$$\begin{aligned} a_i &= -\frac{2c}{N}, \quad i = 1, \ldots, N-1 \\ a_N &= 2c\frac{(N-1)}{N} \end{aligned} \tag{93}$$

One notes that

$$\begin{aligned} \operatorname{tr} I^k &= (N-1)\left(-\frac{2c}{N}\right)^k + \left[2c\frac{(N-1)}{N}\right]^k \\ &= \frac{(2c)^k(N-1)}{N^k}\left[(N-1)^{k-1} + (-1)^k\right] \end{aligned} \tag{94}$$

Thus the matrix I has $N-1$ coincident eigenvalues. But this means that the solutions of eq. (90) are in one-to-one correspondence with the points of a coset $SU(N)/U(1) \times SU(N-1) \approx CP^{N-1}$.

Indeed, the matrix U in formula (91) can be presented in the form

$$U = U_1 V_1 V_2 \tag{95}$$

where

$$V_1 = \begin{pmatrix} & & 0 \\ & \tilde{V}_1 & \vdots \\ & & 0 \\ 0 \ldots 0 & & 1 \end{pmatrix}, \quad \tilde{V}_1 \in SU(N-1) \tag{96}$$

$$V_2 = \exp\{diag(t, \ldots, t, -(N-1)t\}$$

and a matrix U_1 represents an element of the coset $SU(N)/U(1) \times SU(N-1)$. So, due to coincidence of the first $N-1$ eigenvalues of the matrix I, one finds

$$I = U_1 diag\left(-\frac{2c}{N}, \ldots, -\frac{2c}{N}, 2c\frac{(N-1)}{N}\right) U_1^\dagger \tag{97}$$

Therefore the set of solutions of eq. (90) is isomorfic to coset $SU(N)/U(1) \times SU(N-1)$.

Further, using identities (87), (88), one finds that on the surface $\xi^\dagger\xi = c = const$

$$\begin{aligned} D_\mu\xi^\dagger D_\nu\xi &= \frac{1}{8c^2}\,\mathrm{tr}\, I D_{[\mu} I D_{\nu]} I && (98) \\ \xi^\dagger F_{\mu\nu}\xi &= \frac{1}{2}\,\mathrm{tr}\, I F_{\mu\nu} && (99) \end{aligned}$$

where

$$D_\mu I = \partial_\mu I - i[A_\mu, I] \tag{100}$$

Now let us insert in (82) an identity

$$1 = \prod_{x\in\Sigma} \int \prod_a dI^a(x)\delta(\xi^\dagger(x)\lambda^a\xi(x) - I^a(x)) \tag{101}$$

By virtue of (98), (99), the eq. (82) can be rewritten in the form

$$Z = \int D\mu(I) \exp\left\{-i \int\limits_\Sigma dx^\mu \wedge dx^\nu \left[\frac{1}{8c^2}\,\mathrm{tr}\, I D_{[\mu} I D_{\nu]} I - \frac{1}{4}\,\mathrm{tr}\, I F_{\mu\nu}\right]\right\} \tag{102}$$

where

$$D\mu(I) = \left(\prod_a DI^a\right) \int D^2\xi\delta(\xi^\dagger\xi - 1 - \frac{N}{2}) \prod_a \delta(\xi^\dagger\lambda^a\xi - I^a) \tag{103}$$

The consideration given above shows that $D\mu(I)$ is nothing but $SU(N)$-invariant measure on the coset $SU(N)/U(1) \times SU(N-1)$

The explicit form of the measure $D\mu(I)$ is rather cumbersome but, fortunately, it appears that measure $D\mu(I)$ in (102) can be replaced by the measure

$$D\mu'(I) \equiv (DI^a) \prod_{k=2}^{N} \delta(\mathrm{tr}\, I^k - c_{k,N}) \tag{104}$$

where numbers $c_{k,N}$ are defined by formula (94) in which one must put $2c = N+2$ by virtue of the first δ-function in eq.(103).

Indeed, numbers $\mathrm{tr}\, I^k$ define uniquely characteristic equation for the matrix I and, consequently, its eigenvalues. This leads to representation (97). So on the surface $\mathrm{tr}\, I^k = c_{k,N}$ any functional $\Phi(I)$ is invariant under transformation

$$\Phi(I) \to \Phi(V_1 V_2 I V_2^\dagger V_1^\dagger) \tag{105}$$

where a matrix $V_1 V_2$ is defined by eqs. (96). Therefore in any integral

$$\int D\mu'(I)\Phi(I) \tag{106}$$

one can perform all integrations except those corresponding to integration over coset $SU(N)/U(1) \times SU(N-1)$:

$$\int D\mu'(I)\Phi(I) = const \int D\mu(I)\Phi(U_1 diag(-\frac{2c}{N}, \ldots, -\frac{2c}{N}, 2c\frac{N-1}{N})U_1^\dagger) \tag{107}$$

(see (97)). But this just means, in particular, that one can replace the measure $D\mu(I)$ in (102) by the measure $D\mu'(I)$ defined by eq. (104).

Now we can formulate our final result:

$$\begin{aligned} &\mathrm{tr}\ \mathrm{P}\exp\left\{ i \oint_\gamma dx^\mu\, A_\mu \right\} \\ &= \int DI \prod_{k=2}^{N} \delta(\mathrm{tr}\, I^k - c_{k,N}) \\ &\quad \exp\left\{ i \int_\Sigma dx^\mu \wedge dx^\nu \left[-\frac{1}{2(N+2)^2}\, \mathrm{tr}(ID_\mu I D_\nu I) + \frac{1}{4}\, \mathrm{tr}\, IF_{\mu\nu} \right] \right\} \end{aligned} \tag{108}$$

where

$$c_{k,N} = \frac{(N-1)(N+2)^k}{N^k}\left[(N-1)^{k-1} + (-1)^k\right] \tag{109}$$

Let us compare our results with those due to Dyakonov and Petrov [21]. Putting in (108) $N = 2$, $I = I^a\sigma^a$ and the changing $I^a \to 2I^a$, we get:

$$\begin{aligned} &\mathrm{tr}\ \mathrm{P}\exp\left\{i\oint_\gamma dx^\mu\, A_\mu\right\} \\ &= \int DI\delta(I^2-1)\exp\left\{-\frac{i}{2}\int dx^\mu \wedge dx^\nu \left(\varepsilon^{abc}I^aD_\mu I^bD_\nu I^c - I^aF^a_{\mu\nu}\right)\right\} \end{aligned} \tag{110}$$

Comparing formulae (110) and (14), we see that they differ by the factor 1/2 in front of the "action". Most likely this discrepancy arises because of some subtleties in the definition of the functional integrals that play the important role in our discussion. In particular, if one ignores the difference between functional integrals constructed by means of Weyl and normal symbols, one obtains the following representation instead of (80):

$$Z = \int D^2\xi \prod_{x\in\gamma}\delta\left(\xi^\dagger(x)\xi(x) - 1\right) e^{-\oint_\gamma dx^\mu\,\xi^\dagger D_\mu\xi(x)} \tag{111}$$

Further, using formulae (98) and (94) with $c = 1$ instead of $c = 1 + \frac{N}{2}$, one can easy trace that representation (111) leads exactly to Dyakonov-Petrov formula (14). But representation (111) is wrong. So just formula (110) must be considered as correct version of non-Abelian Stokes theorem for $N = 2$.

This doesn't mean, however, that Dyakonov-Petrov formula (14) is incorrect. But it means that the definition of functional integral in (14) must be clarified.

Another discrepancy between our results and those due to Dyakonov and Petrov arises in the case $N \geq 3$. Dyakonov and Petrov pointed out in their work [21] that integration in the functional integral representation for Wilson loop must be performed over the coset $SU(N)/[U(N)]^{N-1}$ whereas in our representation (108) integration is carried out, in fact, over the coset $SU(N)/U(1)\times SU(N-1)$. But nowadays it is hard to discuss this discrepancy because no explicit formulae for the case $N \geq 3$ were given in [21].

5 Bosonic worldline path integral representation for fermionic determinants and Green functions in Euclidean space

Bosonic worldline path integral representation for fermionic determinants can be obtained directly from formulae (8), (10) and results of the section 2:

$$\begin{aligned}
&\ln\det(i\hat{\nabla}+im)\\
&= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\int_{PBC} DqD^2\psi D^2 z\delta\left(z^\dagger z-1-\frac{N}{2}\right)\delta\left(\psi^\dagger\psi-3\right)\\
&\quad \mathcal{N}_T\exp\left\{\int_0^1 dt\left[-\frac{\dot{q}^2}{4T}-z^\dagger\dot{z}-\psi^\dagger\dot{\psi}+i\dot{q}z^\dagger A_\mu z\right.\right.\\
&\quad \left.\left.-T(\psi^\dagger\sigma^{\mu\nu}\psi)(z^\dagger F_{\mu\nu}z)\right]\right\} \qquad (112)
\end{aligned}$$

Here variables $z=\{z^r,\quad r=1,\ldots,N\}$ and $\psi=\{\psi^i,\ i=1,,2,3,4\}$ describe colour and spin degrees of freedom respectively, $z^\dagger$ and $\psi^\dagger$ are complex conjugated to z and ψ,

$$D^2 z \equiv \prod_t\prod_{r=1}^N d(\mathrm{Re}\, z^r(t))\, d(\mathrm{Im}\, z^r(t)) \qquad (113)$$

$$D^2 \psi \equiv \prod_t\prod_{r=1}^N d(\mathrm{Re}\, \psi^i(t))\, d(\mathrm{Im}\, \psi^i(t)) \qquad (114)$$

and $\mathcal{N}_T$ is a normalization constant. The latter can be evaluated from the condition

$$< x|\,\mathrm{tr}\, e^{-T(-\nabla_\mu\nabla^\mu+\sigma^{\mu\nu}F_{\mu\nu})}|x>\Big|_{A=0} = 4N < x|e^{-T\partial_\mu\partial_\mu}|x> = \frac{N}{2\pi^2T^2} \qquad (115)$$

Indeed, putting $A=0$ in (112) and comparing the result with (115), one obtains:

$$
\begin{aligned}
\mathcal{N}_T^{-1} &= \left(\frac{N}{2\pi^2T^2}\right)^{-1} \int\limits_{q(0)=q(1)=x} Dq \int_{PBC} D^2\psi D^2 z \delta\left(z^\dagger z - 1 - \frac{N}{2}\right) \\
& \quad \delta\left(\psi^\dagger\psi - 3\right) \exp\left\{\int_0^1 dt \left[-\frac{\dot{q}^2}{4T} - z^\dagger \dot{z} - \psi^\dagger\dot{\psi}\right]\right\}
\end{aligned} \tag{116}
$$

Obviously, $\mathcal{N}_T$ doesn't depend on x.

Remind, that in Euclidean space $\psi^\dagger\psi$ is $SO(4)$ scalar. So representation (112) is manifestly relativistic and gauge invariant.

Our next task is to derive bosonic worldline path integral representation for Euclidean Green functions. In what follows, we restrict ourselves only to derivation of such representation for generating functional $Z(j)$ for vacuum correlators

$$
< \psi^\dagger_{f_1}(x_1)\psi_{f_1}(x_1)\ldots\psi^\dagger_{f_n}(x_n)\psi_{f_n}(x_n) > \tag{117}
$$

However, our method is quite general and can be easily applied to derivation of analogues representations for arbitrary Green functions.

Standard functional integral representation for $Z(j)$ can be written as

$$
Z(j) = \int DA\, e^{S_{YM}} \int D\Psi D\bar{\Psi} e^{\left\{\sum_f \int dx\, \left[i\bar{\Psi}_f\hat{\nabla}\Psi_f + im_f\bar{\Psi}_f\Psi_f - ij_f\bar{\Psi}_f\Psi_f\right]\right\}} \tag{118}
$$

Integrating with respect to fermionic fields, we get

$$
Z(j) = \int DA\, e^{S_{YM}} \prod_f \det(i\hat{\nabla} + im_f - ij_f) \tag{119}
$$

So our task is reduced to derivation of bosonic path integral representation for determinant

$$
\det(i\hat{\nabla} + im - ij)
$$

The latter problem can be easily solved by applying of the results obtained in the section 2. Indeed,

$$\ln\det(i\hat{\nabla}+im-ij) = \frac{1}{2}\ln[\det(i\hat{\nabla}+im-ij)\gamma^5]^2$$
$$= \frac{1}{2}\ln\det(\nabla_\mu\nabla^\mu - \sigma^{\mu\nu}F_{\mu\nu} + \hat{\partial}j + (m-j)^2)$$
$$= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\,\mathrm{tr}\, e^{-T(-\nabla_\mu\nabla^\mu+\sigma^{\mu\nu}F_{\mu\nu}-\hat{\partial}j+2mj-j^2)} \qquad (120)$$

Using representation for path ordered exponent from section 2, one obtains:

$$\ln\det(i\hat{\nabla}+i(m+j))$$
$$= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\int_{PBC} DqD^2\psi D^2z\delta\left(z^\dagger z - 1 - \frac{N}{2}\right)\delta\left(\psi^\dagger\psi - 3\right)$$
$$\exp\left\{\int_0^1 dt\left[-\frac{\dot{q}^2}{4T} - z^\dagger\dot{z} - \psi^\dagger\dot{\psi} + i\dot{q}z^\dagger A_\mu z\right.\right.$$
$$\left.\left.-T(\psi^\dagger\sigma^{\mu\nu}\psi)(z^\dagger F_{\mu\nu}z - \psi^\dagger\gamma^\mu\psi\partial_\mu j(q) - 2mj(q) + j^2(q)\right]\right\} \qquad (121)$$

Substituting (120) in (121) for each f, we get worldline bosonic path integral representation for generating functional $Z(j)$. The corresponding formulae for n-point correlators are rather cumbersome but quite computable. Author hopes that they can be used for computer simulations on the lattice.

6 Bosonic worldline path integral representation for fermionic determinants and Green functions in Minkowski space and quasi-classical approximation in QCD

The derivation of bosonic worldline path integral representation for fermionic determinants in Minkowski space is slightly more involved then one in Euclidean space. The origin of complications is non-unitarity of finite dimensional representation of the Lorentz group.

The analog of the representation (8) in Minkowski space can be written as

$$\ln\det(i\hat{\nabla}-m)=-\frac{1}{2}\int_0^\infty\frac{dT}{T}\,e^{-im^2T}\,\mathrm{tr}\,e^{iT(-\nabla_\mu\nabla^\mu+\sigma^{\mu\nu}F_{\mu\nu})} \tag{122}$$

Trace in (122) can be represented as functional integral:

$$\mathrm{tr}\,e^{iT(-\nabla_\mu\nabla^\mu+\sigma^{\mu\nu}F_{\mu\nu})}=\int_{PBC}Dq\ \mathrm{tr}\ \mathrm{P}\,e^{i\int_0^1dt\left(-\frac{\dot{q}^2}{4T}+\dot{q}A(q)+T\sigma^{\mu\nu}F_{\mu\nu}(q)\right)} \tag{123}$$

eq. (123) is an analog of eq. (10). Path ordering in (123) corresponds to colour and spinor structures.

However, in contrast to Euclidean case, we cannot directly use the representations of the type (40), (46) to write ordered exponent in (123) as functional integral. Indeed, those representation comprise, in particular, the factor

$$\delta(\psi^\dagger\psi-3) \tag{124}$$

(see (112)) that is not Lorentz invariant because spinor representations of Lorentz group are not unitary.

To obtain manifestly Lorentz invariant representation, we will use, at first, representation (40) for describing of colour degrees of freedom and representation (41) for describing of spinor ones. In such terms eq. (123) can be rewritten as

$$\begin{aligned}&\mathrm{tr}\,e^{i(-\nabla_\mu\nabla^\mu+\sigma^{\mu\nu}F_{\mu\nu})}\\ &\quad=\lim_{\epsilon\to+0}\int_{PBC}DqD^2zD\bar{\psi}D\psi D\lambda\delta(z^\dagger z-1-\frac{N}{2})\\ &\qquad\mathcal{N}_T\exp\left\{i\int_0^1dt\left[-\frac{\dot{q}^2}{4T}+iz^\dagger z+i\bar{\psi}\dot{\psi}+\right.\right.\\ &\qquad\left.\left.z^\dagger\dot{q}A(q)z+T(\bar{\psi}\sigma^{\mu\nu}e^{-\epsilon\frac{d}{dt}}\psi)(z^\dagger F^{\mu\nu}z)+\lambda(\bar{\psi}e^{-\epsilon\frac{d}{dt}}\psi-1)\right]\right\}\end{aligned} \tag{125}$$

In eq. (125) the measure D^2z is defined by (113) but ψ and $\bar{\psi}$ are independent complex variables. $\mathcal{N}_T$ is a normalization constant that will be computed later.

Integrating over $\bar{\psi}$, ψ in (125), one gets:

$$\begin{aligned}
&\mathrm{tr}\, e^{i(-\nabla_\mu\nabla^\mu+\sigma^{\mu\nu}F_{\mu\nu})} \\
&= \lim_{\epsilon\to+0}\int_{PBC} DqD^2zD\lambda\delta(z^\dagger z-1-\frac{N}{2}) \\
&\quad \mathcal{N}_T\mathrm{det}^{-1}\left[-\frac{d}{dt}+iT\sigma^{\mu\nu}(z^\dagger F_{\mu\nu}z)e^{-\epsilon\frac{d}{dt}}+i\lambda e^{-\epsilon\frac{d}{dt}}\right] \\
&\quad \exp\left\{i\int_0^1 dt\left[-\frac{\dot q^2}{4T}+iz^\dagger\dot z+z^\dagger\dot q A(q)z-\lambda\right]\right\}
\end{aligned} \tag{126}$$

But

$$\begin{aligned}
&\lim_{\epsilon\to+0}\mathrm{det}^{-1}\left[-\frac{d}{dt}+iT\sigma^{\mu\nu}(z^\dagger F_{\mu\nu}z)e^{-\epsilon\frac{d}{dt}}+i\lambda e^{-\epsilon\frac{d}{dt}}\right] \\
&= e^{-2i\int_0^1 dt\,\lambda(t)}\mathrm{det}^{-1}\left[-\frac{d}{dt}+iT\sigma^{\mu\nu}(z^\dagger F_{\mu\nu}z)+i\lambda\right]
\end{aligned} \tag{127}$$

by virtue of identity (73). Further,

$$\begin{aligned}
&\mathrm{det}^{-1}\left[-\frac{d}{dt}+iT\sigma^{\mu\nu}(z^\dagger F_{\mu\nu}z)+i\lambda\right] \\
&= \mathrm{det}^{-1}\left[-\gamma^0\frac{d}{dt}+\gamma^0 iT\sigma^{\mu\nu}(z^\dagger F_{\mu\nu}z)+i\gamma^0\lambda\right]
\end{aligned} \tag{128}$$

The operator in R.H.S. of (128) is *anti-Hermitean*. So we can write

$$\begin{aligned}
&\mathrm{det}^{-1}\left[-\gamma^0\frac{d}{dt}+\gamma^0 iT\sigma^{\mu\nu}(z^\dagger F_{\mu\nu}z)+i\gamma^0\lambda\right] \\
&= \int_{PBC} D^2\psi e^{i\int_0^1 dt(i\psi^\dagger\gamma^0\dot\psi+T(\psi^\dagger\gamma^0\sigma\mu\nu\psi)(z^\dagger F_{\mu\nu}z)+\lambda\psi^\dagger\gamma^0\psi)}
\end{aligned} \tag{129}$$

In the last formula $\psi^\dagger$ and ψ are already complex conjugate variables, measure $D^2\psi$ is defined by eq. (114), and the "action" is real. So functional integral (129) is well defined.

Introducing standard notations $\bar\psi=\psi^\dagger\gamma^0$ and substituting (127)-(129) in (126), one obtains, after integration with respect to λ,

$$
\begin{aligned}
& \operatorname{tr} e^{iT(-\nabla_\mu \nabla^\mu + \sigma^{\mu\nu} F_{\mu\nu})} \\
& = \int_{PBC} Dq D^2\psi D^2 z \delta\left(z^\dagger z - 1 - \frac{N}{2}\right) \delta\left(\bar{\psi}\psi - 3\right) \\
& \quad \mathcal{N}_T \exp\left\{ i \int_0^1 dt \left[-\frac{\dot{q}^2}{4T} + i z^\dagger z + i \bar{\psi}\dot{\psi} + \right.\right. \\
& \quad \left.\left. z^\dagger \dot{q} A(q) z + T(\bar{\psi}\sigma^{\mu\nu}\psi)(z^\dagger F^{\mu\nu}(q) z) \right] \right\} \qquad (130)
\end{aligned}
$$

The normalization constant $\mathcal{N}_T$ can be computed in the same way as its analog in eq. (112):

$$
\begin{aligned}
\mathcal{N}_T^{-1} &= \left(-\frac{N}{2\pi^2 T^2}\right)^{-1} \int\limits_{q(0)=q(1)=x} Dq \int_{PBC} D^2\psi D^2 z \delta\left(z^\dagger z - 1 - \frac{N}{2}\right) \\
& \quad \delta\left(\bar{\psi}\psi - 3\right) \exp\left\{ i \int_0^1 dt \left[-\frac{\dot{q}^2}{4T} + i z^\dagger \dot{z} + i \psi^\dagger \dot{\psi} \right] \right\} \qquad (131)
\end{aligned}
$$

Substituting (130) in (122), we obtain desired representation for fermionic determinant:

$$
\begin{aligned}
& \ln \det(i\hat{\nabla} - m) \\
& = -\frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-im^2 T} \int_{PBC} Dq D^2\psi D^2 z \delta\left(z^\dagger z - 1 - \frac{N}{2}\right) \delta\left(\bar{\psi}\psi - 3\right) \\
& \quad \mathcal{N}_T \exp\left\{ i \int_0^1 dt \left[-\frac{\dot{q}^2}{4T} + i z^\dagger z + i \bar{\psi}\dot{\psi} + \right.\right. \\
& \quad \left.\left. z^\dagger \dot{q} A(q) z + T(\bar{\psi}\sigma^{\mu\nu}\psi)(z^\dagger F^{\mu\nu}(q) z) \right] \right\} \qquad (132)
\end{aligned}
$$

The representation (132) is manifestly gauge and Lorentz invariant and comprises only bosonic variables. The "action" in (132) is real. So, as we will see soon, it is convenient for application of stationary phase method.

Now let us derive bosonic path integral representation for generating functional $Z(j)$ of gauge invariant Green functions

$$
G_{f_1,\ldots,f_n}(x_1,\ldots,x_n) = < T(\bar{\psi}_{f_1}(x_1)\psi_{f_1}(x_1)\ldots\bar{\psi}_{f_n}(x_n)\psi_{f_n}(x_n)) > \qquad (133)
$$

The derivation is completely analogous to one given in the previous section for corresponding Euclidean correlators.

For $Z(j)$ there exist standard path integral representation via anticommuting variables:

$$
\begin{aligned}
Z(j) &= \int DAe^{iS_{YM}} \int D\bar{\Psi} D\Psi e^{i\int dx\, \sum_f \left[\bar{\Psi}_f (i\hat{\nabla} - m_f)\Psi_f + j_f \bar{\Psi}_f \Psi_f\right]} \\
&= \int DAe^{iS_{YM}} \prod_f \det(i\hat{\nabla} - m_f + j_f)
\end{aligned} \tag{134}
$$

Repeating with minor changes the derivation of (121), one gets

$$
\begin{aligned}
&\ln\det(i\hat{\nabla} - m - j) \\
&= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-im^2T} \int_{PBC} DqD^2\psi D^2z\delta\left(z^\dagger z - 1 - \frac{N}{2}\right)\delta\left(\bar{\psi}\psi - 3\right) \\
&\quad \mathcal{N}_T \exp\left\{ i\int_0^1 dt\left[-\frac{\dot{q}^2}{4T} + iz^\dagger z + i\bar{\psi}\dot{\psi} + z^\dagger \dot{q}A(q)z + \right.\right. \\
&\quad \left.\left. T(\bar{\psi}\sigma^{\mu\nu}\psi)(z^\dagger F^{\mu\nu}(q)z) + 2mTj(q) - iT\bar{\psi}\gamma^\mu\psi\partial_\mu j(q) - Tj^2(q)\right]\right\}
\end{aligned} \tag{135}
$$

2

Substituting (135) in (134), we obtain worldline pat integral representation for $Z(j)$.

Our next task is the investigation of quasiclassical approximation in QCD. To this end, we will formulate a scheme of evaluation of two-point function

$$
G_{f_0}(x,y) =< T(\bar{\psi}_{f_0}(x)\psi_{f_0}(x)\bar{\psi}_{f_0}(y)\psi_{f_0}(y)) > \tag{136}
$$

This scheme can be easily generalized for evaluation of arbitrary Green functions.

The equations for the stationary point will be interpreted as quasiclassical equations in QCD. They will be formulated in terms of particles that have spin and colour degrees of freedom and interecting with Yang-Mills field.

First of all, we introduce more condenced notations:

$$\begin{aligned}
Q_f &\equiv \{q_f, z_f^\dagger, z_f, \bar{\psi}_f, \psi_f, T_f, m_f^2\} && (137)\\
\int DQ_f(\cdots) &\equiv -\frac{1}{2}\int_0^\infty \frac{dT_f}{T_f}\int_{PBC} DqD^2\psi D^2 z\delta\left(z^\dagger z - 1 - \frac{N}{2}\right)\\
&\quad \delta\left(\bar{\psi}\psi - 3\right)\mathcal{N}_T(\cdots) && (138)\\
S[Q_f, A] &= \int_0^1 dt\left[-\frac{\dot{q}_f^2}{4T_f} + iz_f^\dagger \dot{z}_f + i\bar{\psi}_f\dot{\psi}_f + z_f^\dagger \dot{q}A(q)z_f\right.\\
&\quad \left.+T_f(\bar{\psi}_f\sigma^{\mu\nu}\psi_f)(z_f^\dagger F_{\mu\nu}(q)z_f) - T_f m_f^2\right] && (139)
\end{aligned}$$

Further, using eq. (135), one can get:

$$\frac{\delta}{\delta j(x)}\ln\det(i\hat{\nabla} - m_f + j)\Big|_{j=0} = \int DQ_f e^{iS[Q_f,A]}R(x|Q_f) \qquad (140)$$

$$\begin{aligned}
&\frac{\delta^2}{\delta j(x)\delta j(y)}\ln\det(i\hat{\nabla} - m_f + j)\Big|_{j=0}\\
=& \int DQ_f e^{iS[Q_f,A]}R(x|Q_f)R(y|Q_f)\\
+& i\delta(x-y)\int DQ_f e^{iS[Q_f,A]}T_f\int_0^1 dt_1 dt_2\,\delta(q_f(t_1) - q_f(t_2)) \qquad (141)
\end{aligned}$$

where

$$R(x|Q_f) = T\int_0^1 dt[2m_f - i\bar{\psi}_f(t)\gamma^\mu\psi_f(t)\partial_\mu]\delta(x - q_f(t)) \qquad (142)$$

For any functional $W(j)$

$$\frac{\delta^2 W(j)}{\delta j(x)\delta j(y)} = W(j)\left[\frac{\delta^2\ln W(j)}{\delta j(x)\delta j(y)} + \frac{\delta\ln W(j)}{\delta j(x)}\frac{\delta\ln W(j)}{\delta j(y)}\right] \qquad (143)$$

Using (134), (140), (141) and applying (143) for $W(j) = \det(i\hat{\nabla} - m_{f_0} + j)$, one obtains:

$$G_{f_0}(x-y) = G_{f_0}^{(1)}(x-y) + G_{f_0}^{(2)}(x-y) \tag{144}$$

where $G_{f_0}^{(1)}$ and $G_{f_0}^{(2)}$ correspond to the first and to the second terms in R.H.S. of (143) respectively:

$$\begin{aligned} G_{f_0}^{(1)}(x-y) &= \int DADQ_{f_0}\, R(x,Q_{f_0})R(y,Q_{f_0}) \\ &\quad \exp\left\{ iS_{YM} + iS[Q_{f_0},A] + \sum_f \int DQ_f\, e^{iS[Q_f,A]} \right\} \\ &\quad + \delta(x-y)(\cdots) \qquad (145) \\ G_{f_0}^{(2)}(x-y) &= \int DADQ_{f_0} D{Q'}_{f_0}\, [R(x,Q_{f_0})R(y,Q_{f_0}) \\ &\quad \exp\left\{ iS_{YM} + iS[Q_{f_0},A] + iS[Q^{'}_{f_0},A] \right. \\ &\quad \left. + \sum_f \int DQ_f\, e^{iS[Q_f,A]} \right\} \qquad (146) \end{aligned}$$

In (145) $(\cdots)$ means the factor at $\delta(x-y)$ in R.H.S. of (141).

At first, we investigate the function $G_{f_0}^{(1)}$.

The function $G_{f_0}(x-y)$, in itself, is defined up to counterterm

$$const\,\delta(x-y)$$

by virtue of ultraviolet divergences. Then the last term in R.H.S. of (145) only redefines this counterterm and so can be omitted.

Expanding

$$\exp\left\{ \sum_f \int DQ_f\, e^{iS[Q,A]} \right\} \tag{147}$$

in series, we can represent (145) as

$$G_{f_0}^{(1)}(x-y)$$

$$
\begin{aligned}
&= \sum_{n_f} \frac{1}{\prod_f n_f!} \int DADQ_{f_0} \left(\prod_f \prod_{j_f=1}^{n_f} DQ_{j_f} \right) R(x, Q_{f_0}) R(y, Q_{f_0}) \\
&\quad \exp \left\{ iS_{YM} + iS[Q_{f_0}, A] + \sum_f \sum_{j_f=1}^{n_f} iS[Q_{j_f}, A] \right\} \\
&\equiv \sum_{n_f} G^{(1)}_{f_0}(x-y; \{n_f\}) \qquad (148)
\end{aligned}
$$

Obviously, each function $G^{(1)}_{f_0}(x-y; \{n_f\})$ correspond to contribution of all diagrams comprising n_{f_1} quark loops of flavor f_1, n_{f_2} quark loops of flavor f_2, etc.

We will investigate each term in the series (148) separately. At first we consider the term with $n_f = 0$:

$$
G^{(1)}_{f_0}(x-y; \{n_f = 0\}) = \int DADQ_{f_0}\, e^{iS_{YM} + iS[Q_{f_0}, A]} R(x, Q_{f_0}) R(y, Q_{f_0}) \qquad (149)
$$

The function $R(x|Q_f)$ can be represented as

$$
R(x|Q_f) = \int d^4p \int_0^1 dt\, \tilde{R}(p, t|Q_f) e^{ip(x - q_f(t))} \qquad (150)
$$

where

$$
\tilde{R}(p, t|Q_f) = T(2m_f + p_\mu \bar{\psi}(t) \gamma^\mu \psi(t)) \qquad (151)
$$

Then we change variables:

$$
q(t) \to q'(t) = q(t) - q_0, \quad A(x) \to A(x - q_0) \qquad (152)
$$

where the function $q'(t)$ obeys the boundary conditions

$$
q'(0) = q'(1) = 0 \qquad (153)
$$

By virtue of translational invariance we get

$$
\begin{aligned}
&G^{(1)}_{f_0}(x-y; \{n_f = 0\}) \\
&\quad = \int DA \int_{q(0)=q(1)=0} DQ_{f_0} \int dq_0\, e^{iS_{YM} + iS[Q_{f_0}, A]}
\end{aligned}
$$

$$\int d^4p_1 d^4p_2 \int_0^1 dt_1 \int_0^1 dt_2\, \tilde{R}(p_1,t_1|q_{f_0})R(p_2,t_2|q_{f_0})$$
$$e^{ip_1(y-q_{f_0}(t_1))}e^{ip_2(y-q_{f_0}(t_2))}e^{-i(p_1+p_2)q_0} \tag{154}$$

Integration over q_0 gives $\delta(p_1+p_2)$, and we obtain the following representation for Fourier transformation of $G^{(1)}_{f_0}(x-y;\{n_f=0\})$:

$$\begin{aligned}
&G^{(1)}_{f_0}(p;\{n_f=0\})\\
&\equiv \int dx\, e^{-ipx} G^{(1)}_{f_0}(x;\{n_f=0\})\\
&= \int_0^1 dt_1 \int_0^1 dt_2 \int DA \int_{q_{f_0}(0)=q_{f_0}(1)=0} DQ_{f_0}\, \tilde{R}(p,t_1|Q_{f_0})\tilde{R}(-p,t_2|Q_{f_0})\\
&\quad \exp\{iS_{YM}+iS[Q_{f_0},A]-ip(q_{f_0}(t_1)-q_{f_0}(t_2))\}
\end{aligned} \tag{155}$$

Now it is already easy to write equations for stationary point for the action in (155). Omitting the index f_0, we get:

$$\begin{aligned}
\frac{1}{g^2}\nabla_\nu F^{a\mu\nu} &= \int_0^1 dt\, I^a(t)\dot{q}^\mu(t)\delta(x-q(t))\\
&\quad +T\nabla_\nu\left[\int_0^1 dt\, I^a(t)S^{\mu\nu}\delta(x-q(t))\right]
\end{aligned} \tag{156}$$

$$i\left(\frac{d}{dt}-i\dot{q}^\mu A_\mu(q)\right)z + TS^{\mu\nu}F_{\mu\nu}z=0 \tag{157}$$

$$i\frac{d}{dt}\psi + TF^a_{\mu\nu}I^a\sigma^{\mu\nu}\psi=0 \tag{158}$$

$$\frac{1}{T}\ddot{q}_\mu + \dot{q}^\nu F^a_{\mu\nu}I^a + \nabla_\mu F^a_{\nu\rho}(q)I^a S^{\nu\rho} = p_\mu(\delta(t_1)-\delta(t_2)) \tag{159}$$

$$\frac{1}{4T^2}\int_0^1 dt\,\dot{q}^2 + \frac{1}{2}\int_0^1 dt\, S^{\mu\nu}F^a_{\mu\nu}I^a = m^2 \tag{160}$$

where

$$I^a = z^\dagger\lambda^a z, \quad S^{\mu\nu}=\bar{\psi}\sigma^{\mu\nu}\psi \tag{161}$$

Eqs. (156)-(159) can be derived by variation of the action in (155) with respect to $A^a_\mu,\ z,\ \psi$, and q^μ. In the derivation of (159) we used (156)-(158). The eq. (160) is obtained by differentiation with respect to T.

It easy to obtain closed system of equations in terms of $A^a_\mu,\ z,\ I^a$, and $S^{\mu\nu}$.

Let

$$I = \lambda^a I^a \tag{162}$$

Then

$$iDI = TS^{\mu\nu}[F_{\mu\nu}, I] \tag{163}$$

$$\frac{d}{dt}S^{\mu\nu} = 2TI^a F^{a[\mu}{}_\rho S^{\nu]\rho} \tag{164}$$

where

$$DI \equiv \frac{dI}{dt} - i[\dot{q}^\mu A_\mu(q), I] \tag{165}$$

Unknown functions in (156)-(159) also obey boundary conditions

$$q(0) = q(1) = 0, \quad z(0) = z(1), \quad \psi(0) = \psi(1) \tag{166}$$

They also satisfy the equations

$$z^\dagger z = 1 + \frac{N}{2}, \quad \bar{\psi}\psi = 3 \tag{167}$$

because of the presence of δ-functions in the definition (138) of the measure DQ. Apropos, we didn't introduce Lagrange multipliers to take into account these δ-functions because the solutions of eqs. (157), (158) automatically satisfy the conditions

$$z^\dagger z = const, \quad \bar{\psi}\psi = const \tag{168}$$

Equations (156), (159), and (163) are nothing but generalization of well-known Wong's equations [24] that describe classical spinless particle interacting with Yang-Mills field. Indeed, if one omits the terms containing the tensor of spin $S^{\mu\nu}$ in eqs. (156), (159), and (163) one gets just Wong's equations up to term

$$-\,p_\mu(\delta(t_1)-\delta(t_2)) \tag{169}$$

Eqs. (156)-(158) admit simple interpretation. At "time" t_1 quark–anti-quark pair with momentum p is created. Then quark and anti-quark move interacting with Yang-Mills field. The union of quark and anti-quark trajectories forms a closed loop passing the point $q = 0$. (See (166)). At the "time" t_2 the quark–anti-quark pair annihilates.

An analog of eqs. (156)-(160) for terms with $n_f \neq 0$ in expansion (148) can be derived in the similar way. Let $S^{\{n_f=0\}}$ be the action in the formula (155) and $J^{a\mu}[Q]$ be the current in R.H.S. of eq. (156). Then the analog of $S^{\{n_f=0\}}$ for the term with non zero numbers $\{n_f\}$ in (148) is

$$S^{\{n_f\}} = S^{\{n_f=0\}} + \sum_f \sum_{j_f=1}^{n_f} S[Q_{j_f}, A] \tag{170}$$

So instead of (156) we have in this case equations

$$\frac{1}{g^2}\nabla_\nu F^{a\mu\nu} = J^{a\mu}[Q_{f_0}] + \sum_f \sum_{j_f=1}^{n_f} J^{a\mu}[Q_{j_f}] \tag{171}$$

Equations

$$\frac{\delta S^{\{n_f\}}}{\delta \psi_{j_f}} = 0, \quad \frac{\delta S^{\{n_f\}}}{\delta z_{j_f}} = 0, \quad \frac{\delta S^{\{n_f\}}}{\delta T_f} = 0 \tag{172}$$

coincide in form with (158), (160), and (161), whereas the equation

$$\frac{\delta S^{\{n_f\}}}{\delta q_{j_f}} = 0 \tag{173}$$

differs from (159) by the term (169).

We see that quasiclassical configurations that give the main contribution in functional integrals (148) are defined by equations of very similar structure. The same statement is valid for semiclassical configuration that give the main contribution in the function $G^{(2)}(x-y)$ as well as in any other gauge invariant Green functions. Therefore proposed scheme seems to be sufficiently general for application in QCD and other gauge theories.

In this paper we restrict ourselves to formulation of general scheme of quasiclassical approximation in QCD leaving elaborating of details as

well as applications for forthcoming papers. So at this point we stop our investigation of quasiclassical approximation in QCD. Brief discussion of possible applications will be given in the next section.

7 Conclusion

In the present paper we derived, first, new path integral representations for path ordered exponent (see eqs. (38)-(41), (42), (46), (65)). We give two alternative, entirely independent derivation of these representations. So these results seems quite reliable.

Then we applied these representations to derive new variant of non-Abelian Stokes theorem. Our result is represented by formula (82). Then we transformed the latter to obtain another formulation of non-Abelian Stokes theorem that is similar to one proposed recently by Dyakonov and Petrov [21]. As a result, we got corrected and generalized version of the theorem proved in [21].

Dyakonov-Petrov version of non-Abelian Stokes theorem was already used in discussion of the role of monopoles in QCD [21] and in attempts to derive string-like effective action in framework of QCD [25]. So one can hope that our more general and simple version of this theorem will be also useful in discussion of various problems of QCD.

Further, we derived pure bosonic worldline path integral representations for fermionic determinants as well as fermionic Green functions in Euclidean QCD. (See eqs. (112), (120), and (121)).

This representations comprise only integrations with respect to bosonic variables. On a finite lattice all integrals are quite simple and well convergent. (Remind, that domain of integration with respect to z and ψ in eqs. (112), (120), and (121) is bounded). So representation derived seem quite appropriate for lattice simulations.

Our results for fermionic determinant and Green functions can be used also in another way.

Namely, if one substitute instead of Wilson loop some phenomenological anzatz, one obtain formulation of the theory purely in terms of point particles. The most well-known example of such anzatz is Wilson area law:

$$< \mathrm{tr}\ \mathrm{P} \exp \left\{ i \oint dx^\mu \, A_\mu \right\} >= \exp \left\{ -K S_{min} + \begin{bmatrix} \text{perturbative} \\ \text{corrections} \end{bmatrix} \right\} \quad (174)$$

This anzatz was applied, in particular, to derivation from QCD a quark–anti- quark potential used in potential models (see, for instance, recent papers [26], [27] and references therein).

Other anzatz for Wilson loops were proposed in framework of Dual QCD model [28] and stochastic vacuum model [29]. (See also [27] for comparison of results obtained in framework of these models). Recently a sting-like expression in spirit of stochastic vacuum model was derived in papers [25], [30].Anzatz of another type, that gives an expression for Wilson loop in terms of trajectories of monopoles, was proposed in [21].

All these results could be combined with ours to derive some effective action in terms of point particles that correctly describe colour and spin properties of quarks. Obviously, that such theory is much more simpler than initial quantum field theory and so thus approach of investigation seems to be rather perspective.

Finally, in the present paper we also derived bosonic worldline path integral representation for fermionic determinants and Green functions in Minkowski space and started the investigation of the quasiclassical approximation in QCD.

The key point in the formulation of quasiclassical approximation is quasiclassical QCD equations (156)-(160) and (170)-(173) which arise naturally when one applies the stationary point method to evaluation of functional integrals that defines Green functions in QCD. Quasiclassical equations derived appear to be nothing but a generalization of well-known Wong's equations.

We formulated only those quasiclassical equations which arise in the problem of evaluation of the concrete Green function (136). However, our method is quite general and one can easy to derive analogous equations in generic case.

The next problem in investigation of quasiclassical approximation in QCD is the solution of quasiclassical equation of motion. Though these equations are very complicated, this problem doesn't seem hopeless, at least, in non-relativistic approximation. In electrodynamics in non-relativistic approximation one can neglect bremsstrahlung and retarded effects and, as a result, reformulate initial theory in terms of particles interacting by means of Coulomb forces. In the same way one may hope to derive potential of interaction of heavy quarks in QCD. Such approach is alternative to ones based on various anzatz for Wilson loops.

Another interesting possibility to understand the quark confinement in framework of quasiclassical approach is connected with existence of classical solutions of Yang-Mills equations with singularity on the sphere. Such solution were discussed in the context of the problem of confinement recently in the papers [31] though the existence of such solutions was pointed out by several authors in 70's [32]. Some solutions with singularity on the torus and cylinder was discussed in papers [33].

If solutions with singularity on closed spacelike surface existed also for eqs. (156)-(160) and their modifications, then quarks, moving inside such surface and interacting with Yang-Mills field, couldn't cross the surface. This would mean the confinement of quarks. So it is interesting to investigate singular solutions of equations (156)-(160) and, if such solutions exist, to develop the corresponding quasiclassical perturbative theory.

Appendix A

In this appendix we shall prove the validity of the identity (62).
We will prove (62) by induction. For $N = 2$ it is easy to prove (62) by direct calculations because both sides of (62) are simple rational functions of variables $\exp\{\frac{i}{2}\alpha_r\}$.

Let us denote

$$x_r = e^{i\alpha_r} \tag{A1}$$

Then L.H.S. of (62) can be represented as

$$(-2i)^{N-1} \frac{\prod\limits_{r=1}^{N} \sqrt{x}_r}{\prod\limits_{1\le p<q\le N} (x_p - x_q)} \sum_{j=1}^{N} (-1)^{j+1} x_j^N \prod_{\substack{r,s\neq j \\ 1\le r<s\le N}} (x_r - x_s) \tag{A2}$$

whereas R.H.S. of (62) as

$$(-2i)^{N-1} \left(\prod_{k=1}^{N} \sqrt{x}_k \right) \sum_{j=1}^{N} x_j \tag{A3}$$

So eq. (62) is equivalent to algebraic identity

$$\sum_{j=1}^{N}(-1)^{j+1}x_j^N \prod_{\substack{r,s\neq j\\ 1\leq r<s\leq N}}(x_r-x_s)=\left(\sum_{j=1}^{N}x_j\right)\prod_{1\leq r<s\leq N}(x_r-x_s) \qquad (A4)$$

To prove (A4), we check, at first, that L.H.S. of (A4) is vanished if $x_i = x_j$. Obviously, it is sufficient to consider the case $x_1 = x_2 \equiv x$.

If $x_1 = x_2$ then only $j = 1$ and $j = 2$ terms survive in L.H.S. of (A4). The $j = 1$ term is

$$x_1^N \prod_{2\leq r<s\leq N}(x_r-x_s)=x^N\left(\prod_{p=3}^{N}(x-x_p)\right)\left(\prod_{3\leq p<q\leq N}(x_p-x_q)\right) \qquad (A5)$$

whereas the $j = 2$ term is equal to

$$-x_2^N \prod_{\substack{r,s\neq 2\\ 1\leq r<s\leq N}}(x_r-x_s)=-x^N\left(\prod_{p=3}^{N}(x-x_p)\right)\left(\prod_{1\leq p<q\leq N}(x_p-x_q)\right) \qquad (A6)$$

So terms with $j = 1$ and $j = 2$ are cancelled.

Thus L.H.S. of (A4) can be represented as

$$P(x_1,\ldots,x_N)\prod_{1\leq r<s\leq N}(x_r-x_s) \qquad (A7)$$

We must prove that

$$P(x_1,\ldots,x_N)=\sum_{r=1}^{N}x_r \qquad (A8)$$

L.H.S. of eq. (A4) is polynomial of degree N with respect to each variable x_j. Consequently, polynomial $P(x_1,\ldots,x_N)$ is linear in each x_j. So

$$P(x_1,\ldots,x_N)=a(x_2,\ldots,x_N)x_1+b(x_2,\ldots,x_n) \qquad (A9)$$

Comparing coefficient at x_1^N in L.H.S. of (A4) and (A7), one find that

$$a(x, \ldots, x_N) = 1 \tag{A10}$$

and so it remains to prove that

$$b(x_2, \ldots, x_N) = \sum_{j=2}^{N} x_j \tag{A11}$$

To prove (A11), let us compare the sums of x_1-independent terms in L.H.S. of (A4) and in (A7). They can be written as

$$\sum_{j=2}^{N} (-1)^{j+1} x_j^N \prod_{\substack{2 \le r \le N \\ r \ne j}} (-x_r) \prod_{\substack{2 \le r < s \le N \\ r,s \ne j}} (x_r - x_s) \tag{A12}$$

and

$$b(x_2, \ldots, x_N) \prod_{2 \le r \le N} (-x_r) \prod_{2 \le r < s \le N} (x_r - x_s) \tag{A13}$$

respectively.

So (A11) is equivalent to

$$\sum_{j=2}^{N} (-1)^j x_j^{N-1} \prod_{\substack{r,s \ne j \\ 2 \le r < s \le N}} (x_r - x_s) = \left(\sum_{j=2}^{N} x_j \right) \prod_{2 \le r < s \le N} (x_r - x_s)$$

But the latter equation is valid by virtue of induction assumption. Indeed, it can be obtained from (A4) by change $N \to N-1$, $, x_1 \to x_2, \ldots, x_{N-1} \to x_N$. This finishes the proof of identity (62).

Appendix B

In this appendix we prove the formula (73). First, one notes that up to inessential factor

$$\det \left(-\frac{d}{dt} + iBe^{-\epsilon \frac{d}{dt}} \right) = \det \left(-\frac{d}{dt} e^{\epsilon \frac{d}{dt}} + iB \right) \tag{B1}$$

So it is sufficient to prove that

$$\lim_{\epsilon\to+0} \ln\det\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}+iB\right)-\ln\det\left(-\frac{d}{dt}+iB\right)=\frac{i}{2}\int_0^1 dt\,\mathrm{tr}\,B \quad (B2)$$

or, equivalently,

$$\lim_{\epsilon\to+0} \mathrm{tr}\ln\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}+iB\right)-\mathrm{tr}\ln\left(-\frac{d}{dt}+iB\right)=\frac{i}{2}\int_0^1 dt\,\mathrm{tr}\,B \quad (B3)$$

Let $\hat{\Pi}$ is orthogonal projector in $L^2(S^1)$ on any *finite* dimensional subspace. Then

$$\lim_{\epsilon\to+0}\left[\mathrm{tr}\,\hat{\Pi}\ln\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}+iB\right)-\mathrm{tr}\,\hat{\Pi}\ln\left(-\frac{d}{dt}+iB\right)\right]=0 \quad (B4)$$

In particular, let $\mathcal{H}_1$ be subspace in $L^2(S^1)$ that is orthogonal to one dimensional subspace spanned on the function $\phi_0(t)\equiv 1$. Then

$$\lim_{\epsilon\to+0}\left[\mathrm{tr}\ln\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}+iB\right)-\mathrm{tr}\ln\left(-\frac{d}{dt}+iB\right)\right]$$
$$=\lim_{\epsilon\to+0}\left[\mathrm{tr}_{\mathcal{H}_1}\ln\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}+iB\right)-\mathrm{tr}_{\mathcal{H}_1}\ln\left(-\frac{d}{dt}+iB\right)\right] \quad (B5)$$

In the space $\mathcal{H}_1$ the operator

$$-\frac{d}{dt}e^{\epsilon\frac{d}{dt}} \quad (B6)$$

is invertible. Let $G_\epsilon(t-t')$ is the kernel of

$$\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}\right)^{-1}$$

The function $G_\epsilon(t-t')$ can be expresseed in terms of eigenfunctions

$$\phi_n(t)=e^{2\pi int},\quad n\neq 0 \quad (B7)$$

of the operator (B6) and its eigenvalues

$$\lambda_n = -2\pi i n e^{2\pi i n\epsilon}, \quad n \neq 0$$

Obviously,

$$G_\epsilon(t-t') = \sum_{n\neq 0} \frac{\phi_n(t)\bar{\phi_n}(t')}{\lambda_n} = -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2\pi n(t-t'-\epsilon)}{n} \qquad (B8)$$

Using the formula

$$\sum_{n=1}^{\infty} \frac{\sin 2\pi n\epsilon}{\pi n} = \frac{1}{2} - \epsilon \qquad (B9)$$

(that is valid for $0 < \epsilon < 1$), one finds that

$$\lim_{\epsilon\to+0} G_\epsilon(0) = \frac{1}{2} \qquad (B10)$$

whereas

$$G_0(0) = 0 \qquad (B10')$$

So

$$G_0(0) \neq \lim_{\epsilon\to+0} G_\epsilon(0) \qquad (B11)$$

But for $-1 < t < 1, \quad t \neq 0$,

$$\lim_{\epsilon\to+0} G_\epsilon(t) = G_0(t) \qquad (B11')$$

We will see soon that R.H.S. of (B2) is not vanished just by virtue of (B11).

Further, up to inessential constant,

$$\mathrm{tr}_{\mathcal{H}_1} \ln\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}} + iB\right) = \mathrm{tr}_{\mathcal{H}_1} \ln\left[1 + i\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}\right)^{-1} B\right]$$

$$= \mathrm{tr}_{\mathcal{H}_1}\left[\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}\right)^{-1} B\right] + \frac{1}{2}\mathrm{tr}_{\mathcal{H}_1}\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}\right)^{-1} B \left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}}\right)^{-1} + \ldots \qquad (B12)$$

By virtue of equation

$$\int_0^1 dt\, G_\epsilon(t-t') = 0 \tag{B13}$$

one can replace $\mathrm{tr}_{\mathcal{H}_1}$ by tr in all terms in the series (B12). So

$$\mathrm{tr}_{\mathcal{H}_1} \ln\left(-\frac{d}{dt}e^{\epsilon\frac{d}{dt}} + iB\right) = i\int_0^1 dt\, G_\epsilon(0)\,\mathrm{tr}\, B(t)$$

$$+\frac{1}{2}\int_0^1 dt \int_0^1 dt'\, G_\epsilon(t-t')G_\epsilon(t'-t)\,\mathrm{tr}\, B(t)B(t') + \ldots \tag{B14}$$

In the limit $\epsilon \to +0$ the first term in R.H.S. of (B14) is equal to

$$\frac{i}{2}\int_0^1 dt\,\mathrm{tr}\, B(t)$$

(see (B11)) whereas the sum of all others coincide with

$$\mathrm{tr}_{\mathcal{H}_1}\left(-\frac{d}{dt} + iB\right)$$

by virtue of (B10'), (B11'). This proves the validity of the equation (B2).

To check the formula (B2), let us consider the simplest case of one dimensional harmonic oscillator. Partion function

$$Z(\beta) = \mathrm{tr}\exp\{-\beta a^\dagger a\} \tag{B15}$$

is known exactly:

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta n} = \frac{1}{1-e^{-\beta}} \tag{B15}$$

On the other hand,

$$Z(\beta) = \lim_{\epsilon\to+0}\int_{PBC} D\bar{z}Dz\, \exp\left\{\int_0^1 dt(-\bar{z}\dot{z} - \beta\bar{z}e^{-\epsilon\frac{d}{dt}}z\right\}$$

$$= \det{}^{-1}\left(-\frac{d}{dt} - \beta e^{-\epsilon\frac{d}{dt}}\right) \tag{B16}$$

Putting in (B2) $B = i\beta$, one gets:

$$\det{}^{-1}\left(-\frac{d}{dt}-\beta e^{-\epsilon\frac{d}{dt}}\right)=e^{\frac{\beta}{2}}\det{}^{-1}\left(-\frac{d}{dt}-\beta\right) \qquad (B17)$$

The latter determinant we have already evaluated in the main text of the paper :

$$\det\left(-\frac{d}{dt}-\beta\right)=const\ \sinh\frac{\beta}{2} \qquad (B18)$$

(see eq. (57) for $N=1,\quad \eta=\epsilon=0$).

Comparing (B16)-(B19), we get

$$Z(\beta)=const\,\frac{e^{\frac{\beta}{2}}}{\sinh\frac{\beta}{2}} \qquad (B19)$$

Evaluating the constant from the condition

$$Z(\infty)=1$$

we reproduce the true answer (B15). We would like to stress that if we had put $\epsilon=0$ in (B16) before evaluating of the functional integral the result would have been wrong.

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